From Petrov-Einstein-Dilaton-Axion to Navier-Stokes equation in anisotropic model

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Abstract

In this paper we generalize the previous works to the case that the near-horizon dynamics of the Einstein-Dilaton-Axion theory can be governed by the incompressible Navier-Stokes equation via imposing the Petrov-like boundary condition on hypersurfaces in the non-relativistic and near-horizon limit. The dynamical shear viscosity η of such dual horizon fluid in our scenario, which isotropically saturates the Kovtun-Son-Starinet (KSS) bound, is independent of both the dilaton field and axion field in that limit.

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I. INTRODUCTION

The correspondence between anti-de Sitter gravity and Conformal Field Theory (AdS/CFT) proposed in [1–4] provides a powerful tool on the connection between gravitational physics in the bulk and hydrodynamics living on its boundary. Since Damour firstly found that the gravitational excitations of the black hole horizon behaved like a fluid[5], the hydrodynamical behavior of gravity has been extensively studied in literature[6–23]. In particular, recent progress on fluid/gravity duality in the context of AdS/CFT has shed more insightful light on relating the Einstein's equation to the Navier-Stokes equation for a general class of spacetime geometries[12, 13, 15]. In this setup the gravitational fluctuations confined in between the horizon and a finite cutoff at radius $r = r_c$, can be mapped into a dual holographic fluid living on the cutoff surface. Traditionally, directly disturbing the bulk metric under the regularity condition of the horizon and fixing the induced metric on the boundary, the correspondence between gravitational dynamics in the bulk and hydrodynamics on its boundary can be constructed successfully in the non-relativistic long-wavelength expansion, and those dual hydrodynamical quantities can be also explicitly read off via the standard procedure in AdS/CFT dictionary, whose dependence on the cutoff r_c is viewed as the renormalization group flow in the fluid[18–23].

Very remarkably, the gravity/fluid duality was firstly implemented by imposing the Petrov-like condition on the cutoff surface in the near horizon limit [24] instead of the regularity condition on the horizon in Rindler spacetime. It has been shown that embedding a hypersurface Σ_c into a Rindler spacetime, the gravitational fluctuation can be reduced exactly to the incompressible Navier-Stokes equation living on one lower dimensional flat spacetime. More explicitly, in this approach keeping the induced metric fixed and taking the extrinsic curvature as fundamental variables, one directly required the extrinsic curvature perturbations to satisfy the Petrov-like condition such that in the non-relativistic limit and the near horizon limit the continuous equation of the Brown-York tensor can give rise to the incompressible Navier-Strokes equation. In this sense, the Petrov-like condition plays an important holographic role on this correspondence. In contrast to traditional approaches, this kind of setup is mathematically much simpler and elegant, since it doesn't even need to construct explicitly the metric perturbation in the bulk, thus no need to solve the perturbed Einstein equations in the bulk either. Due to this powerful condition, recently there have

been greatly interesting extensions in [25–35].

On the other hand, an interesting conjecture, which said that the ratio of dynamical shear viscosity to entropy density was no less than $\frac{1}{4\pi}$, was proposed in [36]. This is the so-called "KSS bound". However, this bound was later found to be violated in the anisotropic holographic plasma[37]. Since then, the problem of KSS bound violation has attracted a great deal of attention[38–43] in the anisotropic gravitational systems.

Motivated by the above progress, it should be interesting to ask how about the gravity/fluid duality under the Petrov-like boundary condition in the context of an anisotropic gravitational system. In this paper we will provide an answer to this question. It turns out that we can still obtain the standard Navier-Stokes equations under the non-relativistic and near horizon limit, but the anisotropy of the gravitational background only results in an anisotropy of the background pressure, while the ratio of dynamical viscosity to entropy density is still isotropic and saturates the KSS bound.

The rest of our paper is organized as follows. In Section 2 we briefly review some important formulas. In Section 3 we will derive the incompressible Navier-Strokes equations on a spatially flat hypersurface from an anisotropic gravitational system in detail. In Section 4 we will give a summary and some discussions. In the appendix we present a detailed calculation for the last term of the Petrov-like condition (23).

II. SOME IMPORTANT FORMULAS

In this section, here we would like to review some important relations that play the role of bridge on the gravity/fluid duality. Let us start with the gravitational side. Firstly, we naturally require p+2 dimensional spacetime geometry to satisfy the standard Einstein theory:

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + T_{\mu\nu}, \quad \mu, \nu = 0, \ldots, p+1, \tag{1}$$

where $g_{\mu\nu}$ is a metric of the $p+2$ dimensional spacetime, Λ is a cosmological constant and $T_{\mu\nu}$ is energy momentum tensor in the bulk. Secondly, in order to discuss the behavior of dual fluid, in the $p+2$ dimensional bulk space one needs to embed a $p+1$ dimensional timelike hypersurface Σ_c with a induced metric γ_{ab}, whose extrinsic curvature K_{ab} should

satisfy the $p+1$ "momentum constraints"

$$D^a(K_{ab} - \gamma_{ab}K) = T_{\mu b}n^\mu, \tag{2}$$

as well as the "Hamiltonian constraint"

$${}^{p+1}R + K_{ab}K^{ab} - K^2 - 2\Lambda = -2T_{\mu\nu}n^\mu n^\nu, \tag{3}$$

where D_a is compatible with the induced metric on Σ_c, namely $D_a\gamma_{bc} = 0$, K is the trace of extrinsic curvature and n^μ is the unit normal to Σ_c.

For imposing Petrov-like condition on this cutoff surface one must decompose the p+2 dimensional Weyl tensor into those p+1 dimensional quantities in terms of the intrinsic curvature, extrinsic curvature and induced metric on the hypersurface. The framework has been specifically introduced in previous literature[24–26],

$$\begin{aligned}
C_{abcd} &= {}^{p+1}R_{abcd} + K_{ad}K_{bc} - K_{ac}K_{bd} + \frac{2\Lambda - 2T}{p(p+1)}(\gamma_{ad}\gamma_{bc} - \gamma_{ac}\gamma_{bd}) \\
&\quad - \frac{1}{p}\gamma_a{}^\alpha\gamma_b{}^\beta\gamma_c{}^\gamma\gamma_d{}^\delta(g_{\alpha\gamma}T_{\delta\beta} - g_{\alpha\delta}T_{\gamma\beta} - g_{\beta\gamma}T_{\delta\alpha} + g_{\beta\delta}T_{\gamma\alpha}), \\
C_{abc(n)} &= D_aK_{bc} - D_bK_{ac} - \frac{1}{p}\gamma_a{}^\alpha\gamma_b{}^\beta\gamma_c{}^\gamma n^\delta(g_{\alpha\gamma}T_{\delta\beta} - g_{\alpha\delta}T_{\gamma\beta} - g_{\beta\gamma}T_{\delta\alpha} + g_{\beta\delta}T_{\gamma\alpha}), \\
C_{a(n)c(n)} &= KK_{ac} - K_a{}^bK_{bc} + \gamma_a{}^\alpha\gamma_c{}^\gamma R_{\alpha\gamma} - {}^{p+1}R_{ac} - \frac{2(\Lambda - T)}{p(p+1)}\gamma_{ac} \\
&\quad - \frac{1}{p}\gamma_a{}^\alpha\gamma_c{}^\gamma n^\beta n^\delta(g_{\alpha\gamma}T_{\delta\beta} - g_{\alpha\delta}T_{\gamma\beta} - g_{\beta\gamma}T_{\delta\alpha} + g_{\beta\delta}T_{\gamma\alpha}).
\end{aligned} \tag{4}$$

Thus the Petrov-like boundary condition on Σ_c is defined as

$$C_{(\ell)i(\ell)j} = \ell^\mu m_i{}^\nu \ell^\alpha m_j{}^\beta C_{\mu\nu\alpha\beta} = 0, \tag{5}$$

where $p+2$ Newman-Penrose-like vector fields satisfy the following relations

$$\ell^2 = k^2 = 0, \quad (k,\ell) = 1, \quad (k, m_i) = (\ell, m_i) = 0, \quad (m^i, m_j) = \delta^i{}_j, \tag{6}$$

where $\gamma_{ab} = g_{ab} - n_an_b$, $C_{abc(n)} = C_{abc\mu}n^\mu$. In the absence of matter field, the traceless Petrov-like boundary condition on Σ_c actually causes $p(p+1)/2 - 1$ constraints on the extrinsic curvature such that it can reduce exactly the $(p+1)(p+2)/2$ degrees of freedom of extrinsic curvature to p+2 unconstrained variables which can be viewed as the energy density, pressure and velocity fields of the dual fluid living on the cutoff surface. The Hamiltonian constraint becomes a equation of state linking the energy density to pressure of such dual fluid, while

the $p+1$ momentum constraints govern the evolution of the dynamics of gravity which is regarded as a fluid living on hypersurface.

In the presence of matter field, on the surface we generally need further to introduce some appropriate boundary condition for matter field so that the total degree of freedom can correctly present the dual hydrodynamical behavior. For vacuum case of Einstein theory, it can be governed by the initial-boundary value problem (IBVP) [44]. Based on the idea by Friedrich and Nagy, we can see that Petrov-like boundary condition can be viewed as the free boundary data of IBVP of vacuum Einstein system. This hint gives us a guideline for searching a suitable boundary condition for matter field.

III. NAVIER-STOKES EQUATIONS IN THE ANISOTROPIC SPACETIME

In this section, employing the Petrov-like boundary condition on the cutoff surface embedded in a five-dimensional anisotropic spacetime, we will explicitly demonstrate how to derive the incompressible Navier-Stokes Equations from the anisotropic linear axion model under the near horizon and non-relativistic limit. As showed in [39, 45, 46], the action of the Einstein-Dilaton-Axion theory can be written as

$$S = \int d^5x\sqrt{-g}[\frac{1}{2\kappa^2}(R+12) - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2], \tag{7}$$

where $2\kappa^2 = 16\pi G_5$ is the five-dimensional gravitational coupling, ϕ and χ are the dilaton field and the axion field, respectively. Here we have set the cosmological constant scale $L=1$. Then the equations of motion for the axion, the dilaton and the gravitational field can be presented respectively

$$\nabla_\mu(e^{2\phi}\nabla^\mu\chi) = 0, \tag{8}$$

$$\nabla_\mu\nabla^\mu\phi - e^{2\phi}(\partial\chi)^2 = 0, \tag{9}$$

$$R_{\mu\nu} - \partial_\mu\phi\partial_\nu\phi - e^{2\phi}\partial_\mu\chi\partial_\nu\chi + 4g_{\mu\nu} = 0. \tag{10}$$

Here we have used $\kappa^2 = 8\pi G_5 = 1$. We consider such spacetime geometry[39, 45, 46] preserving rotational invariance in the x-y plane, which can be written as

$$ds_5^2 = e^{-\frac{\phi}{2}}[-f(r)B(r)dt^2 + 2\sqrt{B(r)}drdt + r^2dx^2 + r^2dy^2 + r^2H(r)dz^2], \tag{11}$$

where ϕ, B, H and $f(r) \equiv r^2F(r)$ only depend on the radial coordinate r. χ is a linear function, namely $\chi = az$, where a is a constant. At the horizon of this geometry $f(r_h)$

should be vanishing, namely $f(r_h)=0$, which is equivalent to $F(r_h)=0$. $H(r)$ is related to the dilaton field ϕ, namely $H(r)=e^{-\phi(r)}$. When $H(r)=1$, this spacetime has spatial isotropy. Otherwise, it is anisotropic. The relevant anisotropic solutions have appeared in [39, 45, 46]. Thus here we do not try to repeat it in detail, whereas mainly focus on the hydrodynamical behavior of the gravity. The hypersurface Σ_c located by $r=r_c$ outside the horizon of this geometry can be naturally introduced by

$$\begin{aligned} ds_4^2 &= -f(r_c)B(r_c)e^{-\frac{1}{2}\phi(r_c)}dt^2+r_c^2e^{-\frac{1}{2}\phi(r_c)}dx^2+r_c^2e^{-\frac{1}{2}\phi(r_c)}dy^2+r_c^2H(r_c)e^{-\frac{1}{2}\phi(r_c)}dz^2 \\ &= -dx^{0^2}+r_c^2e^{-\frac{1}{2}\phi(r_c)}dx^2+r_c^2e^{-\frac{1}{2}\phi(r_c)}dy^2+r_c^2H(r_c)e^{-\frac{1}{2}\phi(r_c)}dz^2, \end{aligned} \tag{12}$$

where $\sqrt{f(r_c)B(r_c)e^{-\frac{1}{2}\phi(r_c)}}t=x^0$. Obviously the hypersurface embedded is intrinsic flat. To exhibit explicitly the non-relativistical behavior of dual hydrodynamics on this hypersurface, we further need to introduce a parameter λ by rescaling the time coordinate $\lambda x^0=\tau$. Thus the above induced metric can be rewritten as

$$ds^2{}_{p+1}=-\frac{1}{\lambda^2}d\tau^2+r_c^2e^{-\frac{1}{2}\phi(r_c)}dx^2+r_c^2e^{-\frac{1}{2}\phi(r_c)}dy^2+r_c^2H(r_c)e^{-\frac{1}{2}\phi(r_c)}dz^2, \tag{13}$$

Later, one can see that both the non-relativistical limit and the near horizon limit will be implemented by identifying the parameter λ with the location of the hypersurface, namely $r_c-r_h=(\alpha\lambda)^2$ such that taking $\lambda\to 0$ means these limits can be achieved simultaneously. Taking $\lambda\to 0$ limit implies that the hypersurface is highly accelerated, which is thought of as the large mean curvature. It is easily checked that the background quantities of the components of extrinsic curvature defined on the hypersurface Σ_c have following forms in the coordinate(τ,x^i)

$$\begin{aligned} K^{\tau(B)}{}_\tau &= e^{\frac{1}{4}\phi_c}[\frac{f_c'}{2\sqrt{f_c}}+\frac{\sqrt{f_c}B_c'}{2B_c}-\frac{\sqrt{f_c}\phi_c'}{4}] \\ K^{\tau(B)}{}_i &= 0 \\ K^{2(B)}{}_2 &= K^{1(B)}{}_1=e^{\frac{1}{4}\phi_c}(\frac{\sqrt{f_c}}{r_c}-\frac{\sqrt{f_c}\phi_c'}{4}) \\ K^{3(B)}{}_3 &= e^{\frac{1}{4}\phi_c}(\frac{\sqrt{f_c}}{r_c}-\frac{\sqrt{f_c}\phi_c'}{4}+\frac{\sqrt{f_c}H_c'}{2H_c}) \\ K &= e^{\frac{1}{4}\phi_c}[\frac{f_c'}{2\sqrt{f_c}}+\frac{\sqrt{f_c}B_c'}{2B_c}+\frac{3\sqrt{f_c}}{r_c}-\sqrt{f_c}\phi_c'+\frac{\sqrt{f_c}H_c'}{2H_c}], \end{aligned} \tag{14}$$

where the prime $'$ denotes derivative with respect to r. Here, for the convenience, we have abbreviated the background terms, namely $\Phi(r_c)\equiv\Phi_c$, where $\Phi(r_c)$ includes $f(r_c)$, $\phi(r_c)$,

$B(r_c)$ and $H(r_c)$ above. Furthermore, in order to describe definitely the perturbation effect of gravity, as appearing in [25], we still take the Brown-York stress tensor as fundamental variable, which is defined as

$$t^a{}_b = \delta^a{}_b K - K^a{}_b, \tag{15}$$

In coordinate(τ, x^i), we can further rewrite the components of extrinsic curvature and its trace in terms of their corresponding Brown-York tensors ,

$$K^\tau{}_\tau = \frac{t}{3} - t^\tau{}_\tau, \qquad K^\tau{}_i = -t^\tau{}_i, \qquad K^i{}_j = -t^i{}_j + \delta^i{}_j \frac{t}{3}, \qquad K = \frac{t}{3}. \tag{16}$$

Now we can start to investigate the hydrodynamical behavior of the gravity on the cutoff surface in the near horizon limit and non-relativistic limit. Note that in contrast to the conventional perturbation method that uses the metric expansion to solve the perturbation Einstein equations, and then governs the Brown-York tensor dynamical behavior identified with the energy momentum tensor of hydrodynamics on hypersurface, here we take the Brown-York tensors as the fundamental variables and consider directly its fluctuations on the cutoff surface, without solving the perturbation gravitational equations, while keeping the intrinsic induced metric of the surface fixed. Thus we can expand their components in powers of λ as

$$\begin{aligned}
t^\tau{}_i &= 0 + \lambda t^\tau{}_i{}^{(1)} + \dots \\
t^\tau{}_\tau &= e^{\frac{1}{4}\phi_c}[\frac{3}{r_c}\sqrt{f_c} - \frac{3\sqrt{f_c}\phi'_c}{4} + \frac{\sqrt{f_c}H'_c}{2H_c}] + \lambda t^\tau{}_\tau{}^{(1)} + \dots \\
t^i{}_j &= e^{\frac{1}{4}\phi_c}[\frac{f'_c}{2\sqrt{f_c}} + \frac{\sqrt{f_c}B'_c}{2B_c} + \frac{2\sqrt{f_c}}{r_c} - \frac{3\sqrt{f_c}\phi'_c}{4} + \frac{\sqrt{f_c}H'_c}{2H_c} - \delta^i{}_3\delta^3{}_j\frac{\sqrt{f_c}H'_c}{2H_c}]\delta^i{}_j + \lambda t^i{}_j{}^{(1)} + \dots \\
t &= 3e^{\frac{1}{4}\phi_c}[\frac{f'_c}{2\sqrt{f_c}} + \frac{\sqrt{f_c}B'_c}{2B_c} + \frac{3\sqrt{f_c}}{r_c} - \sqrt{f_c}\phi'_c + \frac{\sqrt{f_c}H'_c}{2H_c}] + \lambda t^{(1)} + \dots,
\end{aligned} \tag{17}$$

where the Latin alphabets i, j of the term $\delta^i{}_3\delta^3{}_j\frac{\sqrt{f_c}H'_c}{2H_c}$ inside the square brackets in the third line of the above equations runs as the Latin alphabets i, j outside the square brackets, but the former does not join in other behaviors such as contraction. To obtain the perturbation behavior of gravity in the near horizon limit, we need to expand the background terms of

the above stress tensors around the horizon in powers of $r_c - r_h$ identified with $\alpha^2\lambda^2$.

$$\begin{aligned} f_c &= f'_h\alpha^2\lambda^2 + \frac{f''_h}{2}\alpha^4\lambda^4 + \dots \\ H_c &= H_h + H'_h\alpha^2\lambda^2 + \dots \\ B_c &= B_h + B'_h\alpha^2\lambda^2 + \dots \\ \phi_c &= \phi_h + \phi'_h\alpha^2\lambda^2 + \dots \end{aligned} \tag{18}$$

one will see that this strategy unifying the non-relativistic limit and near horizon limit plays an essential role in deducing successfully the standard Navier-Strokes Equation. Now we consider the specific form of "Hamiltonian constraint" in Eq.(3), whose form is rewritten in terms of the components of Brown-York stress tensor

$$t^\tau{}_\tau t^\tau{}_\tau + t^n{}_m t^m{}_n - \frac{t^2}{3} - \frac{2}{\lambda^2}\gamma^{mn}t^\tau{}_m t^\tau{}_n = -12 - 2T_{\mu\nu}n^\mu n^\nu \tag{19}$$

Note that all the indices of the physical quantities on the hypersurface here are lowered or raised with γ_{ab} and γ^{ab}. Plugging Eq.(18) and Eq.(17) into Eq.(19), we find that the leading term of the constraint at the order λ^{-2} vanishes automatically and the sub-leading one gives rise to

$$t^{\tau(1)}_{\ \tau} = -2e^{-\frac{1}{4}\phi_h}\gamma^{(0)mn}t^{\tau(1)}_{\ m}t^{\tau(1)}_{\ n} - 3e^{\frac{1}{4}\phi_h}(\frac{f'_h}{r_h} - \frac{3f'_h}{4}\phi'_h) + e^{-\frac{1}{4}\phi_h}(12 - \frac{a^2e^{\frac{7\phi_h}{2}}}{r_h^2}) \tag{20}$$

where $\gamma^{(0)mn} \equiv \gamma^{mn}(r_h)$ and $H(r) = e^{-\phi(r)}$ have been used. Now we turn to considering Petrov-like boundary condition on hypersurface. After choosing $3+2$ Newman-Penrose-like vector fields,

$$\sqrt{2}\ell = \partial_0 - n, \quad \sqrt{2}k = -\partial_0 - n, \quad m_1 = \frac{e^{\frac{1}{4}\phi}}{r}\partial_1, \quad m_2 = \frac{e^{\frac{1}{4}\phi}}{r}\partial_2, \quad m_3 = \frac{e^{\frac{1}{4}\phi}}{r\sqrt{H}}\partial_3, \tag{21}$$

the Petrov-like boundary condition (5) can be generally presented as

$$C_{0i0j} + C_{0ij(n)} + C_{0ji(n)} + C_{i(n)j(n)} = 0 \tag{22}$$

To obtain conveniently the dynamical behavior of this geometry, making use of Eq.(4) we can rewrite explicitly the condition in terms of Brown-York stress tensor as

$$\begin{aligned} t^\tau{}_\tau t^k{}_j + \frac{2}{\lambda^2}\gamma^{ki}t^\tau{}_i t^\tau{}_j - t^k{}_i t^i{}_j + 2\lambda\partial_\tau(\frac{t}{3}\delta^k{}_j - t^k{}_j) + (\frac{t}{3})^2\delta^k{}_j \\ -\frac{1}{\lambda}\gamma^{ki}(\partial_j t^\tau{}_i + \partial_i t^\tau{}_j) - t^\tau{}_\tau(\frac{t}{3})\delta^k{}_j + B^k{}_j = 0, \end{aligned} \tag{23}$$

where

$$B^k{}_j \equiv \gamma^{ki}\gamma_i{}^\alpha\gamma_j{}^\beta R_{\alpha\beta} - \frac{1}{3}(\lambda^2\delta^k{}_jT_{\tau\tau} + \delta^k{}_jT_{\alpha\beta}n^\alpha n^\beta - 2\lambda\delta^k{}_jT_{\tau\alpha}n^\alpha). \tag{24}$$

The details of the calculation can be found in the Appendix. Substituting Eq.(17) and Eq.(18) into the above equation (23), we find that the background as a leading term satisfies automatically the Petrov-like condition at the order of $\frac{1}{\lambda^2}$:

$$\frac{f'_h}{4\alpha^2\lambda^2}e^{\frac{1}{2}\phi_h}\delta^k{}_j - \frac{f'_h}{4\alpha^2\lambda^2}e^{\frac{1}{2}\phi_h}\delta^k{}_j = 0, \tag{25}$$

and the perturbations of the gravity as sub-leading terms at the order of λ^0, which gives rise to

$$\begin{aligned} t^{k(1)}_{\ j} &= 2e^{-\frac{1}{4}\phi_h}\gamma^{(0)ki}t^{\tau(1)}_{\ i}t^{\tau(1)}_{\ j} + e^{\frac{1}{4}\phi_h}(\frac{f'_h}{r_h} - \frac{f'_h}{4}\phi'_h - \delta^k{}_3\delta^3{}_j\frac{f'_h}{2}\phi'_h)\delta^k{}_j + \frac{t^{(1)}}{3}\delta^k{}_j \\ &\quad -e^{-\frac{1}{4}\phi_h}\gamma^{(0)ki}(\partial_j t^{\tau(1)}_{\ i} + \partial_i t^{\tau(1)}_{\ j}) + e^{-\frac{1}{4}\phi_h}(-4\delta^k{}_j + \frac{a^2e^{\frac{7}{2}\phi_h}}{r_h^2}\delta^{k3}\delta^3{}_j). \end{aligned} \tag{26}$$

Here we have used $\frac{\sqrt{f'_h}}{\alpha} = 1$. In intrinsic flat hypersurface, since $T_{\mu b}n^\mu$ vanishes for choosing $b = \tau, i$, the momentum constraint(2) in terms of Brown-York tensor $t^a{}_b$ reduces to be

$$\partial_a t^a{}_b = 0. \tag{27}$$

When $b = \tau$, we can directly derive the incompressible condition from the above equation, which is

$$O(\lambda^{-1}): \quad \partial_k v^k = 0. \tag{28}$$

When $b = j$, utilizing Eq.(26), we can straightforwardly derive the standard Navier-Stokes equations,

$$\partial_\tau v_j + v^k\partial_k v_j - \nu\partial^2 v_j + \partial_j P_\perp = 0, \quad (j = 1, 2) \tag{29}$$

$$\partial_\tau v_3 + v^k\partial_k v_3 - \nu\partial^2 v_3 + \partial_3 P_\parallel = 0. \quad (j = 3) \tag{30}$$

Here we have defined the transverse and longitudinal pressures as

$$P_\perp = 2e^{-\frac{1}{4}\phi_h}[\frac{t^{(1)}}{3} + e^{\frac{1}{4}\phi_h}(\frac{f'_h}{r_h} - \frac{f'_h}{4}\phi'_h) - 4e^{-\frac{1}{4}\phi_h}], \tag{31}$$

$$P_\parallel = 2e^{-\frac{1}{4}\phi_h}[\frac{t^{(1)}}{3} + e^{\frac{1}{4}\phi_h}(\frac{f'_h}{r_h} - \frac{3f'_h}{4}\phi'_h) + e^{-\frac{1}{4}\phi_h}(\frac{a^2e^{\frac{7}{2}\phi_h}}{r_h^2} - 4)], \tag{32}$$

respectively. It is worth noting that, in the context of AdS/CFT, the background pressures in the dual fluid contain some important background information in the bulk. The different pressures in the dual Navier-Stokes equations (29) and (30), in some sense, reversely indicate that the dual spacetime is anisotropic, which distinguishes from the situation with identical pressures that mean the dual spacetime is isotropic. In this sense, the background pressures are non-trivial. However, the difference between the transverse pressure $P_\perp$ and the longitudinal one $P_\parallel$, as shown in Eqs.(31) and (32), is just a constant. Moreover, from Eqs.(31) and (32), it is easy to find that the background pressures contribute nothing to the dynamical behavior in the dual hydrodynamics, since their spatial derivatives vanish automatically. As a consequence, we can drop the background pressures terms so that the usual Navier-Stokes equation can be still presented as

$$\partial_\tau \upsilon_j + \upsilon^k \partial_k \upsilon_j - \nu \partial^2 \upsilon_j + \partial_j P = 0, \quad (j = 1, 2, 3) \tag{33}$$

where the pressure P has been identified with $2e^{-\frac{1}{4}\phi_h}\frac{t^{(1)}}{3}$. In addition, we have also identified $t^{\tau}{}_{j}^{(1)} = \frac{1}{2}e^{\frac{1}{4}\phi_h}\upsilon_j$ and the kinematic shear viscosity $\nu = e^{-\frac{1}{4}\phi_h}$ above. In particular, the ratio of dynamical viscosity to entropy density is

$$\frac{\eta}{s} = \frac{\nu\rho}{s} = \frac{\frac{1}{2}e^{\frac{1}{4}\phi_h}e^{-\frac{1}{4}\phi_h}}{\frac{1}{4G}} = 2G = \frac{1}{4\pi}. \tag{34}$$

Here we have used $8\pi G = 1$, and the entropy density $s = \frac{1}{4G}$. The above equation indicates that under the non-relativistic and near-horizon limit the dynamical viscosity of this dual fluid is still isotropic and saturates the KSS bound[36], even in the anisotropic holographic setup considered here.

IV. SUMMARY AND DISCUSSIONS

In this paper we have generalized the previous works[24–26] to the case in which the dynamical behavior of the Einstein-Dilaton-Axion theory can be governed by the incompressible Navier-Stokes equations via imposing the Petrov-like boundary condition on hypersurface in the non-relativistic limit as well as in the near horizon limit, such that the holographic nature and the elegance of the Petrov-like condition have been further disclosed. Here requiring that the Petrov-like condition holds on the cutoff surface, while keeping the induce metric on this surface fixed, we have demonstrated that in contrast with the Navier-Stokes equation with unit kinematic shear viscosity in the previous works, the kinematic

shear viscosity of such fluid equation in our scenario is related to the value of the dilaton field on the horizon. However, the ratio of dynamical shear viscosity to entropy density is still the constant $\frac{1}{4\pi}$, although the anisotropic effect has been considered. This likely means that such boundary condition under the large mean curvature limit sustains the KSS bound. In addition, the anisotropic background spacetime gives rise to the anisotropy of the background pressures, which distinguishes from the isotropic case that leads to the same background pressure. The difference of dual hydrodynamic pressures between the transversal and the longitudinal to the anisotropic direction is only constant which is given by the gradient of the axion field and the relevant values of the dilaton field on the horizon. Since the spatial derivatives of the background pressure terms vanish automatically, they do not affect the dynamical effect of such dual fluid. As a result, we can ignore these constant pressure terms and redefine the pressure in dual hydrodynamics such that the usual Navier-Stokes equation can be still obtained.

In standard approaches in AdS/CFT, the (dynamical) shear viscosity of the boundary dual fluid in anisotropic setups generally becomes a symmetric tensor with different eigenvalues in anisotropic directions, which violates the KSS bound in certain directions[37, 41]. Our result for the horizon fluid is more like that considered in [47, 48], where the ratio of dynamical shear viscosity to entropy density is always $\frac{1}{4\pi}$ in spite of the anisotropy of the holographic setup. Nevertheless, a deep understanding of the relationship between these formalisms is still lacking.

On the other hand, it should be an interesting problem that adopting the traditional non-relativistic long-wavelength limit, how about the hydrodynamic behaviors of gravity at finite cutoff surfaces and the corresponding transport coefficients in such anisotropic systems. The systems at cutoff surfaces can interpolate between the horizon fluid and the boundary CFT, which is then related to the understanding of the different results of shear viscosities mentioned above. This aspect is left for future works.

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Appendix

In this Appendix we need to present that the Ricci tensor and the momentum energy tensor in bulk are how to contribute the Petrov-like condition in detail. From the variation of the action (7), the momentum energy tensor can be given by

$$T_{\mu\nu}=\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi+e^{2\phi}\partial_\mu\chi\partial_\nu\chi-\frac{1}{2}e^{2\phi}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi, \tag{35}$$

where χ is a linear Axion field, namely $\chi=az$ and a is constant. Using the above equation, the all components of this tensor can be straightforwardly calculated as

$$\begin{aligned}
T_{tt} &= -[\frac{1}{2}g_{tt}g^{rr}(\partial_r\phi)^2+\frac{1}{2}e^{\frac{5}{2}\phi}g_{tt}\frac{a^2}{r^2H(r)}], && (36)\\
T_{tr} &= -[\frac{1}{2}g_{tr}g^{rr}(\partial_r\phi)^2+\frac{1}{2}e^{\frac{5}{2}\phi}g_{tr}\frac{a^2}{r^2H(r)}], && (37)\\
T_{ti} &= 0, && (38)\\
T_{rr} &= (\partial_r\phi)^2, && (39)\\
T_{ri} &= 0, && (40)\\
T_{ij} &= -[\frac{1}{2}g_{ij}g^{rr}(\partial_r\phi)^2+\frac{1}{2}e^{\frac{5}{2}\phi}g_{ij}\frac{a^2}{r^2H(r)}-e^{2\phi}a^2\delta^3{}_i\delta^3{}_j], && (41)\\
T &\equiv g^{\mu\nu}T_{\mu\nu}=-[\frac{3}{2}g^{rr}(\partial_r\phi)^2+\frac{3}{2}e^{\frac{5}{2}\phi}\frac{a^2}{r^2H(r)}]. && (42)
\end{aligned}$$

The Einstein equation (1) in the dilaton-axion model can be rewritten as

$$R_{\mu\nu}=\frac{-T-12}{3}g_{\mu\nu}+T_{\mu\nu}. \tag{43}$$

Imposing the above equations, Eq.(24) can be presented as

$$B^k{}_j=-4\delta^k{}_j+e^{\frac{5}{2}\phi_c}\frac{a^2}{r_c^2H_c}\delta^{k3}\delta^3{}_j-\frac{1}{3}\delta^k{}_jf_ce^{\frac{\phi_c}{2}}(\partial_r\phi_c)^2. \tag{44}$$

Here we have used the result

$$T_{\tau\mu}n^\mu=0. \tag{45}$$

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